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ERGODIC TRANSFORMATIONS AND SEQUENCES OF INTEGERS

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ABSTRACT

Using an ergodic transformation defined on an infinite measure space, we discuss complements in $\mathbb{Z}$ of the set $\mathbb{A}$ consisting of finite sums of odd powers of 2.

Let $\mathbb{A}$ and $\mathbb{B}$ be two infinite subsets of the set of non-negative integers $\mathbb{N}$. If every integer $n \in \mathbb{N}$ can be written uniquely as $n = a + b$ with $a \in \mathbb{A}$ and $b \in \mathbb{B}$ then $\mathbb{A}$ and $\mathbb{B}$ are said to be complementing subsets of $\mathbb{N}$, and we write $\mathbb{A} \oplus \mathbb{B} = \mathbb{N}$. The structure of such subsets of $\mathbb{N}$ is well known; see [1] and [7]. However, little is known when $\mathbb{N}$ is replaced by the set of all integers $\mathbb{Z}$. Some progress was made in this direction recently in [2] . It was shown in [2] that the ergodic measure preserving transformation introduced in [6] belongs to a wide class of ergodic measure preserving transformations associated with complementing subsets of $\mathbb{N}$. Namely, if $\mathbb{A}$ is an infinite subset of $\mathbb{N}$ which has an infinite complement $\mathbb{B}$ in $\mathbb{N}$, then it was shown in [2] how to construct an ergodic measure preserving transformation that accepted $\mathbb{A}$ as an exhaustive weakly wandering sequence. Subsequently, using properties of such a transformation it was shown how to construct a continuum number of complements of $\mathbb{A}$ in $\mathbb{Z}$. Nevertheless, the subset $\mathbb{A}$ happens to possess many more complements in $\mathbb{Z}$ than the ones exhibited in [2] . It is possible to characterise all the complements in $\mathbb{Z}$ for such a subset $\mathbb{A}$ in $\mathbb{N}$. This will be done in a subsequent paper, where the important notion, the rank of a number, is introduced and discussed in connection with the subsets $\mathbb{A}$ and $\mathbb{B}$ of $\mathbb{N}$.

In another direction it was shown in [3] that the example in [6] and the ergodic transformations discussed in [2] belong to the class of ergodic measure preserving

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transformations of finite type. Transformations of finite type were introduced and discussed in some detail in [3] ; these are ergodic measure preserving transformations which admit exhaustive weakly wandering sets of finite measure. A set W is a weakly wandering (w.w.) set for a transformation T if there exists an infinite subset A of the integers such that for $a, a' \in \mathsf{A}$ and $a \neq a'$ we have $T^aW \cap T^{a'}W = \phi$, and W is an exhaustive weakly wandering (ex.w.w.) set for T if $X = \cup_{a \in \mathsf{A}} T^aW$ (disj) . The corresponding set or sequence of integers A is also called w.w. or ex.w.w., respectively, for T.

In the general case the collection of all w.w. sequences for a transformation T is clearly an isomorphism invariant, but it is too general to effectively distinguish among ergodic measure preserving transformations defined on an infinite measure space. The collection of all ex.w.w. sequences for a transformation T is also an isomorphism invariant, and in certain situations seems to be more effective in classifying ergodic measure preserving transformations defined on an infinite measure space. The transformations discussed in [2] belong to a class of transformations that possess sufficient regularity properties among transformations of finite type to make it possible for us to classify them more effectively. We leave the discussion of such questions to another paper [4] .

In this article we shall concentrate on the example T discussed in [6] . We associate that example with the sequence of integers A consisting of 0 and all finite sums of odd powers of 2 . The sequence A has the complement B in $\mathbb{N}$ where B consists of 0 and all finite sums of even powers of 2. We discuss some properties of the transformation T and exhibit different ex.w.w. sets for the sequence A. This way we are able to shed more light on the behaviour of the complements C of A in $\mathbb{Z}$. The generalisation of our discussion to complementing subsets of $\mathbb{N}$ in general and to the associated class of transformations discussed in [2] is straight forward and except for notational complications can be carried through without difficulty.

The transformation T and some of its properties

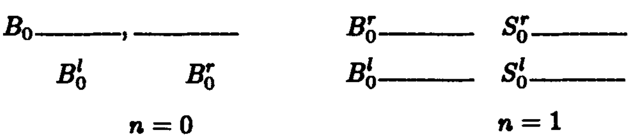
In this section we construct once more the transformation introduced in [6]. We shall build the measure space (X, B, m), which will be isomorphic to the infinite Lebesgue measure space of the real line, and build the transformation T on it as a cutting and stacking construction; see [5]. Next we discuss a few properties of the transformation T, and establish some preliminary results that are needed in the next section. We proceed to the construction of the space (X, B, m) and the transformation T inductively as follows:

Step $n = 0 \quad (n = 2k, k = 0)$.

We start with the Lebesgue measure space of the unit interval $W = [0,1)$. At this step we have a stack, denoted by B_0, which consists of one level of measure 1 . The transformation T is not defined anywhere.

Step $n = 1 \quad (n = 2k+1, k = 0)$.

We cut the stack B_0 vertically into two equal pieces, B_0^l and B_0^r, the left and right halves of B_0, respectively. We place B_0^r on top of B_0^l, and to the right of the resulting stack we add an isomorphic stack. We denote the corresponding pieces of the new stack by S_0^l and S_0^r , respectively. We obtain the following picture:

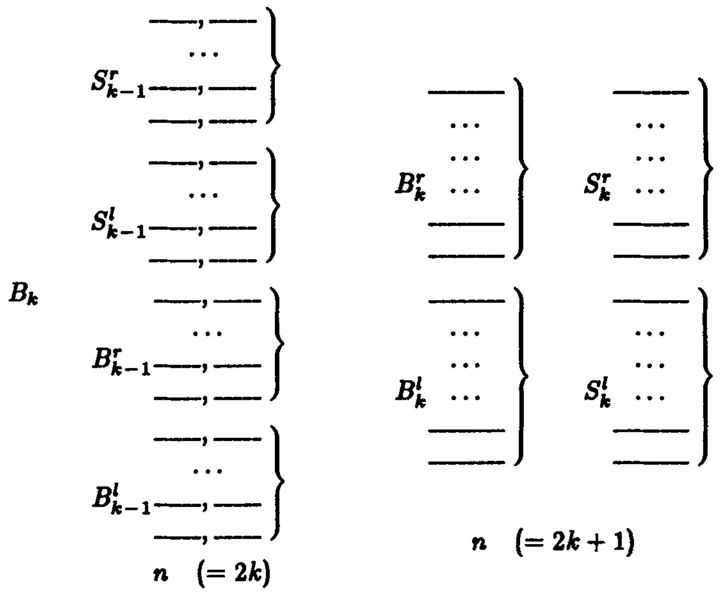


We continue inductively for $n = 2, 3, 4, \ldots$ as follows:

When $n \quad (= 2k)$ is an even integer we place the stack S_{k-1}^l, S_{k-1}^r on top of the stack B_{k-1}^l, B_{k-1}^r. We denote by B_k the resulting stack which consists of 2^n levels each of measure 2^{-k}. We define the transformation T on B_k except on the top level of it by mapping a point onto the corresponding point on the level above it. The transformation T is thus extended from the previous step and is defined everywhere on the stack B_k except on the top level, and T^{-1} is defined everywhere on the stack B_k except on the bottom level.

When $n \quad (= 2k+1)$ is an odd integer, we cut the stack B_k vertically into two equal pieces, B_k^l and B_k^r, the left and right halves of the stack B_k, respectively. We place B_k^r on top of B_k^l, and to the right of the resulting stack we add an isomorphic stack. We denote the corresponding pieces of this new stack by S_k^l and S_k^r, respectively. We obtain the picture at the top of the following page.

Continuing in this way we obtain the measure space (X, B, m) and the ergodic measure preserving transformation T defined everywhere on it. We note that the transformation T^{-1} is not defined at the point 0 , the left hand point of the set W. In what follows we remove from X the point 0 and all its images under the powers of T; then T becomes a $1-1$ onto ergodic measure preserving transformation. We shall also denote by $\mathcal{A}$ the set of integers consisting of 0 and all finite sums of odd powers of 2 , and by $\mathcal{B}$ the set of integers consisting of 0 and all finite sums of even powers of 2.



PROPOSITION 1: *The set $W = [0,1)$ is an ex.w.w. set under the sequence A for the transformation T.*

Proof: In the construction of the transformation T we note that at step $n = 1$ the set $W^1 = W$ covers the left stack, and the set T^2W^1 covers the right stack. At step $n = 3$ the set $W^3 = W^1 \cup T^2W^1$ (disj) covers the left stack, and the set $T^{2^3}W^3$ covers the right stack. More generally, at step $n \geq 3$, for n an odd integer, the set $W^n = W^{n-2}T^{2^{n-2}}W^{n-2}$ (disj) covers the left stack, and the set $T^{2^n}W^n$ covers the right stack. This shows that W is an ex.w.w. set for the transformation T under the sequence A. □

We observe that in the construction of the transformation T and the space X, at step $n = 2k$ for $k = 1, 2, \ldots$, the picture consists of 2^{2k} intervals each of length 2^{-k}. We shall call each of these a dyadic interval D_k of length 2^{-k}. We note that for $k = 0, D_0 = W[= [0,1)]$ is the dyadic interval of length 1 , and it splits into two dyadic subintervals D_1 and TD_1 of length 2^{-1} . Moreover, for each one of these dyadic subintervals D there exists a corresponding dyadic interval of length 2^{-1} ; namely, T^2D. Thus there exist a total of 2^2 dyadic intervals of length 2^{-1}. More generally, for $k > 0$, each of the 2^{2k} dyadic intervals D_k of length 2^{-k} splits into two dyadic subintervals D_{k+1} and $T^{2^{2k}}D_{k+1}$ of length $2^{-(k+1)}$. Moreover, for each one of these dyadic subintervals D there exists a corresponding dyadic interval of length $2^{-(k+1)}$; namely, $T^{2^{2k+1}}D$. Thus there exist a total of $2^{2(k+1)}$ dyadic intervals of length $2^{-(k+1)}$.

For any subset V of X let us denote by $\bar{V} = \cup_{a\in A}T^aV$. We note that, since the sets T^aW for $a \in A$ are mutually disjoint, if D' and D'' are two dyadic subintervals of W, then $D' \cap D'' = \phi$ implies $\bar{D}' \cap \bar{D}'' = \phi$. This fact is also true for any two dyadic subintervals of a dyadic interval D in X.

PROPOSITION 2: *For $k > 0$, let D_k be a dyadic interval of length 2^{-k}, then $T^{2^{2k-1}}\bar{D}_k = \bar{D}_k$.*

Proof: We prove by induction. The set $D_0 = W(= [0,1))$ is the dyadic interval of length 1 . We note that $D_0 = D_1 \cup TD_1$ (disj), where D_1 and TD_1 are the two dyadic subintervals of D_0 of length 2^{-1}. Then $\bar{D}_0 = X$ implies

$$X = \bar{D}_1 \cup T\bar{D}_1(\text{disj}). \tag{1}$$

Applying T to both sides of (1) we get

$$X = T\bar{D}_1 \cup T^2\bar{D}_1(\text{disj}). \tag{2}$$

If we denote by D either of the dyadic subintervals of D_0 then (1) and (2) imply $T^2\bar{D} = \bar{D}$. The additional two dyadic intervals of length 2^{-1} are T^2 images of the corresponding dyadic subintervals of D_0 . This proves the Lemma for all dyadic intervals of length 2^{-1}.

Next we assume that the Proposition is true for all dyadic intervals D_k of length 2^{-k}. Then $D_k = D_{k+1} \cup T^{2^{2k}}D_{k+1}$ (disj), where D_{k+1} and $T^{2^{2k}}D_{k+1}$ are the two dyadic subintervals of D_k of length $2^{-(k+1)}$. This implies

$$\bar{D}_k = \bar{D}_{k+1} \cup T^{2^{2k}}\bar{D}_{k+1} \quad (\text{disj}). \tag{3}$$

Applying $T^{2^{2k}}$ to both sides of (3), since $T^{2^{2k-1}}\bar{D}_k = \bar{D}_k$, we obtain

$$\bar{D}_k = T^{2^{2k}}\bar{D}_{k+1} \cup T^{2^{2k+1}}\bar{D}_{k+1} \quad (\text{disj}). \tag{4}$$

If we denote by D either of the dyadic subintervals of length $2^{-(k+1)}$ of D_k, then combining (3) and (4) we conclude $T^{2^{2k+1}}\bar{D} = \bar{D}$. The additional dyadic intervals of length $2^{-(k+1)}$ are $T^{2^{2k+1}}$ images of the corresponding dyadic subintervals of D_k. □

For two subsets E and F of X and a positive integer $k > 0$ we shall say that E is at least k steps away from F if $T^i E \cap F = \phi$ for $0 < |i| < k$. We note that it is possible for a set E to be k steps away from itself.

PROPOSITION 3: *Let U be an ex.w.w. set under the sequence A for T, and V a subset of U. Let U and V both be the finite union of dyadic intervals; and let $k > 0$ be a positive integer. Then there exists an integer $j > k$ such that the set $E = (U \setminus V) \cup T^j V$ is an ex.w.w. set under the sequence A, and the set $T^j V$ is at least k steps away from E.*

Proof: We note that according to Proposition 1 above and Theorem 1 of [3] the set U is necessarily of measure 1. Since U is a finite union of dyadic intervals, by splitting these intervals further, we represent both sets U and V as a finite union of dyadic intervals each of the same length 2^{-i} for some $i > 0$. We choose i such that $2^{2i} > k$. Once more we refer to the construction of the transformation T and the space X. At step $2i+3$ the set U is contained totally inside the left stack and is contained below 2^{2i+1} levels from the top of the left stack. The same holds true for the set V . We let $j = 2^{2i+3}$; then the set $E = T^j V$ will be the finite union of dyadic intervals, and it will be contained totally inside the right stack and below 2^{2i+1} levels from the top of it. This says that the set $T^j V$ will be at least $2^{2i+1} > k$ steps away from U. It is clear that if we let $E = (U \setminus V) \cup T^j V$ then from Proposition 2 follows that $X = \bar{E}$. Since $m(E) = 1$, then Theorem 1 of [3] implies that E is an ex.w.w. set under the sequence A for T. □

Complements of the set A in $\mathbb{Z}$

For any set of integers E we shall denote by $\mathsf{E} - \mathsf{E}$ the set $\{n \in \mathbb{Z} : n = e - e' \text{ for } e, e' \in \mathsf{E}\}$. In the sequel we normalize the complements C of A in $\mathbb{Z}$ by requiring that $0 \in \mathsf{C}$.

PROPOSITION 4: *Let V be an ex.w.w. set under A for the transformation T. For any point $x \in V$ consider the set $\mathsf{C} = \mathsf{C}_x = \{n \in \mathbb{Z}; T^n x \in V\}$, the hitting times of the point x in V. Then $\mathsf{A} \oplus \mathsf{C} = \mathbb{Z}$.*

Proof: Since V is weakly wandering under A we have $T^{a-a'} V \cap V = \phi$ for $a, a' \in \mathsf{A}, a \neq a'$. Also if $c_1, c_2 \in \mathsf{C}_x$ for $x \in V$, then $c_1 - c_2 \in \mathsf{C}_y$ for $y = T^{c_2} x \in V$. It follows that

$$(1) \qquad (\mathsf{A} - \mathsf{A}) \cap (\mathsf{C} - \mathsf{C}) = \{0\} \quad \text{for any } \mathsf{C} = \mathsf{C}_x, x \in V.$$

Next we fix $x \in V$ and let $\mathsf{C} = \mathsf{C}_x$. Since $X = \cup_{a \in \mathsf{A}} T^a V$ (disj), it follows that for $n \in \mathbb{Z}, T^n x \in T^a V$ for some $a \in \mathsf{A}$. Then $T^{n-a} x \in V$ implies $n - a \in \mathsf{C}$, or $n = a + c$ for some $a \in \mathsf{A}$ and $c \in \mathsf{C}$. The uniqueness of the representation $n = a + c$ follows from (1). □

According to the above Propositions 1 and 4 , for any $x \in W$ it follows that the set $\mathsf{C}_x = \{n \in \mathbb{Z} : T^n x \in W\}$ is a complement of A in $\mathbb{Z}$. Thus it is possible to obtain a continuum number of complements of A in $\mathbb{Z}$. The complements that are obtained in this way however, possess the following additional property described in the next Proposition.

We recall the complement of the set A in $\mathbb{N}$, namely the set B consisting of 0 and all finite sums of even powers of 2.

PROPOSITION 5: *Let* C *be a complement of* A *in* $\mathbb{Z}$*, such that* $\mathsf{C} = \mathsf{C}_x = \{n \in \mathbb{Z} : T^n x \in W\}$ *for some* $x \in W$. *Then* $\mathsf{C} - \mathsf{C} = \mathsf{B} - \mathsf{B}$.

Proof: For any $x \in W$ let $\mathsf{C} = \mathsf{C}_x$ be defined as above. We index the members of C such that $c_0 = 0$ and $c_n < c_{n+1}$ for $n \in \mathbb{Z}$; we also consider the set $\mathsf{D} = \{d_n\}$ where $d_n = c_n - c_{n-1}$ for $n \in \mathbb{Z}$. We note that

$$W = D_1 \cup T D_1 \quad \text{(disj)}, \tag{5.1}$$

where D_1 is a dyadic interval of length $1/2$;

$$D_1 = D_2 \cup T^{2^2} D_2 \quad \text{(disj)}, \tag{5.2}$$

where D_2 is a dyadic interval of length $1/2^2$; and in general for $n \geq 1$

$$D_n = D_{n+1} \cup T^{2^{2n}} D_{n+1} \quad \text{(disj)}, \tag{5.3}$$

where D_{n+1} is a dyadic interval of length $1/2^{n+1}$.

From (5.1) we conclude that every 2nd element of D equals 1, from (5.2) we conclude that every 2^2th element of D equals $2^2 - 2^0$, and from (5.3) we conclude in general that

$$\text{every } 2^n\text{th element of } \mathsf{D} \text{ equals } 2^{2n} - 2^{2n-2} - \cdots - 2^0 \text{ for } n \geq 1. \tag{5.4}$$

If we index the set B such that $b_0 = 0$ and $b_n < b_{n+1}$ for $n \in \mathbb{N}$, and let $\mathsf{D}' = \{d_n : d_n = b_n - b_{n-1}; n \geq 1\}$, then D' also has the same property (5.4) as above. □

There exist complements $\mathbb{C}$ of $\mathbb{A}$ in $\mathbb{Z}$ however, that do not have the property that $\mathbb{C} - \mathbb{C} = \mathbb{B} - \mathbb{B}$.

PROPOSITION 6: *For any positive integer $k > 0$ there exists an ex.w.w. set W' under the set $\mathbb{A}$ such that for $x \in W'$ the set $\mathbb{C}_x = \{n \in \mathbb{Z} : T^n x \in W'\}$ has the property that for $n \neq 0$ if $n \in \mathbb{C}_x - \mathbb{C}_x$ then $|n| > k$.*

Proof: In the construction of the transformation T we note that at step $2k+1$ the set $W = [0,1)$ splits into $p = 2^{k+1}$ dyadic intervals, each one a separate level inside the left stack. We designate these levels $D_0, D_1, \ldots, D_{p-1}$. We let D_0 be the bottom level and note that it is at least k steps away from itself. We keep the set D_0 and inductively choose the integers $j_1 < j_2 < \cdots < j_{p-1}$; we use Proposition 3 repeatedly at each step. First we choose $j_1 > k$ and replace the set D_1 by $T^{j_1} D_1$ so that the set $T^{j_1} D_1$ is at least k steps away from the set $W \cup T^{j_1} D_1$. Next we choose $j_2 > j_1 + k$ and replace the set D_2 by $T^{j_2} D_2$ so that the set $T^{j_2} D_2$ is at least k steps away from the set $W \cup T^{j_1} D_1 \cup T^{j_2} D_2$. We continue this way until we replace each of the sets D_i by the sets $T^{j_i} D_i$ for $0 < i < p$, respectively. We let $W' = D_0 \cup \bigcup_{0<i<p} T^{j_i} D_i$. Then the set W' is ex.w.w. under the sequence $\mathbb{A}$, and W' is at least k steps away from itself. It follows that for $x \in W'$ the set $\mathbb{C}_x = \{n \in \mathbb{Z} : T^n x \in W'\}$ has the stated properties. □

Thus it is possible to construct complements $\mathbb{C}$ of $\mathbb{A}$ in $\mathbb{Z}$ such that $\mathbb{C} - \mathbb{C}$ does not contain a given integer n. We note that all the complements $\mathbb{C}$ constructed so far are of the form $\mathbb{C} = \mathbb{C}_x = \{n \in \mathbb{Z} : T^n x \in W'\}$ for almost all $x \in W'$ for some W' that is ex.w.w. under $\mathbb{A}$ for T. Complements $\mathbb{C}$ of $\mathbb{A}$ in $\mathbb{Z}$ obtained this way however, possess an additional property. To clarify this let $\mathbb{C} = \{c_i | i \in \mathbb{Z}\}$ be a set of integers such that $c_i > c_{i-1}$ for $i \in \mathbb{Z}$. Then the set $\mathbb{C}$ is said to satisfy the block repeat property if for any $i \in \mathbb{Z}$ and any block of $k+1$ integers $c_i < c_{i+1} < \cdots < c_{i+k}$ contained in $\mathbb{C}$ there exists a $j > i$ such that the block of $k+1$ integers $c_j < c_{j+1} < \cdots < c_{j+k}$ contained in $\mathbb{C}$ satisfy $c_{i+p+1} - c_{i+p} = c_{j+p+1} - c_{j+p}$ for $0 < p < k$.

The complements $\mathbb{C}$ of $\mathbb{A}$ in $\mathbb{Z}$ that have been constructed so far by the above methods possess the block repeat property as defined above. This is an immediate consequence of the fact that the transformation T under consideration is an ergodic measure preserving transformation defined on the infinite Lebesgue measure space of the real line. We proceed to show that it is possible to construct complements $\mathbb{C}$ of $\mathbb{A}$ in $\mathbb{Z}$ that do not possess the block repeat property.

We need the following Proposition 7 which is a slightly stronger version of Proposition 6 .

PROPOSITION 7: *Let W' be the finite union of dyadic intervals, and let it be an ex.w.w. set under $\mathbb{A}$ for T. Let $x \in W'$, and consider the set of integers $\mathbb{C}' = \{n \in \mathbb{Z} : T^n x \in W'\}$ and a finite subset $\mathbb{C}_f = \{c_1, \cdots, c_n\}$ of $\mathbb{C}'$. Then for any positive integer $k > 0$ there exists an ex.w.w. set W'' under $\mathbb{A}$ such that $x \in W''$, and the set $\mathbb{C}'' = \{n \in \mathbb{Z} : T^n x \in W''\}$ has the property that $\mathbb{C}_f \subset \mathbb{C}''$, and for $n \in \mathbb{C}'' - \mathbb{C}''$ and $n \notin \mathbb{C}_f - \mathbb{C}''$ we have $|n| > k$.*

Proof: The proof is similar to the proof of Proposition 6 . We let $c_0 = 0$ and note that the points $T^{c_i} x$ for $0 \le i \le n$ belong to the set W'. We choose a large integer $k' > k$ such that at the step $2k' + 1$ in the construction of the transformation T the set W' splits into p dyadic intervals, each one a separate level inside the left stack, and such that the points $T^{c_i} x$ for $0 \le i \le n$ belong to different dyadic intervals. We designate these intervals by $D_0, D_1, \cdots, D_{p-1}$ such that the first $n + 1$ of these intervals $D_0, D_1, \cdots, D_n$, contain the points $x, T^{c_1} x, \cdots, T^{c_n} x$, respectively. Next we chose the integers $j_{n+1} < j_{n+2} < \cdots < j_{p-1}$ such that for each $i, n < i < p$, the set $T^{j_i} D_i$ is at least k steps away from the set $W' \cup T^{j_{n+1}} D_{n+1} \cup \cdots \cup T^{j_i} D_i$. Finally, the set $W'' = D_0 \cup \cdots \cup D_n \cup T^{j_{n+1}} D_{n+1} \cup \cdots \cup T^{j_{p-1}} D_{p-1}$ is an ex.w.w. set under $\mathbb{A}$. It follows that the set $\mathbb{C}'' = \{n \in \mathbb{Z} : T^n x \in W''\}$ has the stated property. □

PROPOSITION 8: *Let $0 < k_1 < k_2 < k_3 \cdots$ be an increasing sequence of positive integers. Then there exists a complement $\mathbb{C}$ of $\mathbb{A}$ in $\mathbb{Z}$, where $\mathbb{C} = \{c_n | n = 0, 1, 2, \cdots\}$ is a sequence of integers with the property that $|c_n| - |c_{n-1}| > k_n$ for all $n \ge 1$.*

Proof: We choose the sequence $\mathbb{C} = \{c_n | n = 0, 1, 2, \ldots\}$ inductively. We let $c_0 = 0$ and enumerate $\mathbb{Z} = \{z_n | n = 0, 1, 2, \ldots\}$ such that $\mathbb{Z} = \{0, 1, -1, 2, -2, \ldots\}$. For $k_1 > 0$ we use Proposition 6, choose a set W_1, and fix the point $x \in W_1$. Then the set $\mathbb{C}_1 = \{n \in \mathbb{Z} : T^n x \in W_1\}$ is a complement of $\mathbb{A}$ in $\mathbb{Z}$; thus there exists a unique $c_1 \in \mathbb{C}_1$ with $a_1 + c_1 = 1 (= z_1)$ for a unique $a_1 \in \mathbb{A}$. We choose c_1 and note that $|c_1| > k_1$. Next for $k_2 > 0$ we use Proposition 7 and choose the set W_2 such that the set $\mathbb{C}_2 = \{n \in \mathbb{Z} : T^n x \in W_2\}$ is a complement of $\mathbb{A}$ in $\mathbb{Z}$ and $c_1 \in \mathbb{C}_2$; thus there exists a unique $c_2 \in \mathbb{C}_2$ where $a_2 + c_2 = -1 \quad (= z_2)$ for a unique $a_2 \in \mathbb{A}$; we choose c_2 and note that $|c_2| - |c_1| > k_2$. We continue by induction. Having chosen the integers $\mathbb{C}_f = \{c_1, \ldots, c_{n-1}\}$ for $k_n > 0$ we use Proposition 7 and choose the set W_n, such that $x \in W_n$, and $C_f \subset C_n$ where $C_n = \{i \in Z : T^i x \in W_n\}$. Finally we choose the unique $c_n \in C_n$ where $c_n + a_n = z_n$ for the unique $a_n \in A$; we note that $|c_n| - |c_{n-1}| > k_n$. Continuing this way the set $C = \{c_n | n = 0, 1, \ldots\}$ has the stated properties. □

From the above discussions it is clear that there are many complements C of A in Z, and moreover, it may seem that there exist little restrictions on these complements. However, we are able to introduce the important notion of the rank or index associated with a number and describe quite fully the structure of these complements. For future reference we mention the rank of an integer n in this case as being the largest power of 2 that divides n. For example, the sum $\{0, 28\} \oplus A$ is direct, i.e. each sum is unique; the same is true for $\{0, 52\} \oplus A$ and $\{0, 28, 52\} \oplus A$. However, using the rank it is possible to show that both $\{0, 28\}$ and $\{0, 52\}$ can be extended to complements of A in Z; but $\{0, 28, 52\}$ cannot be extended to any complement.

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